FUNDAMENTAL CONCEPTS AND METHODS
FOR SYSTEMS MODELING:
A Mathematical Foundation for the Description of
Physical, Chemical, and Biological Processes

Outline of Lecture Unit 8
FINITE AND INFINITE INTEGRAL TRANSFORM METHODS:
FUNDAMENTAL CONCEPTS

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FINITE AND INFINITE INTEGRAL TRANSFORM METHODS:
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1 INTRODUCTION

1.1 Motivation for Study

Integral transform methods provide systematic ways for simplifying and solving ordinary and partial differential equations:

- Finite integral transforms are particularly useful for solving partial differential equation problems; they allow to apply the separation of variables approach in a consistent and systematic manner.

- Infinite (either “simply” or “doubly” infinite) transforms are far more widely used in engineering applications than finite transforms; they allow the transformation of ordinary differential equations into algebraic equations and of partial differential equations into ordinary differential equations.

Integral transforms are one example of the power and generality of inner product space and linear operator methods applied to the analysis of “real world” problems; although they can be introduced “arbitrarily” (and probably one or two of them have, in elementary courses on differential equations) without mentioning function spaces and inner products, the underlying theory provides the rationale and the methodology for developing appropriate integral transforms.

1.2 Comments

- Introduction of transform methods at this point is appropriate since they utilize directly the concepts of inner product spaces and spectral representations of operators; applications will be more extensively considered during the study of solution methods for partial differential equations.

- By far the most popular infinite transforms in engineering applications are the (well known from undergraduate curricula) Laplace and Fourier transforms; however many other transforms have also been extensively studied (e.g., Mellin, Hankel, Radon, etc.) and are appropriate for specific applications. In general the “geometry” of the system, the boundary conditions, etc., determine which transform is the “best” for the problem at hand.
2 GENERAL INTRODUCTION OF INTEGRAL TRANSFORMS

2.1 General Definition

Consider a function vector space with inner product (typically a Hilbert space) with a real or complex associated scalar field and let \( t \) be the independent variable for any vector \( f(t) \) in this space. An (integral) transform \( \tilde{f}(s) \) of \( f(t) \) is defined by selecting an appropriate kernel function \( k(t, s) \), through:

\[
\tilde{f}(s) \text{ or } \tilde{f}(\bar{s}) = (k(t, s) \cdot f(t))
\]

So, the inner product maps vectors from the original space to a new function space with independent variable \( s \).

It should be kept in mind at all times that integral transforms are linear operators on vector spaces themselves (...that are used to simplify other operators by acting both on them and the vectors of the space).

2.2 Determining the Appropriate Kernel for a Transform

The utility of the above general definition of integral transforms stems from the fact that it can be applied directly to any linear operator \( L \) defined on the function space under consideration and utilize the concepts of adjoint operators and eigenvectors to find kernel functions that “work” with this operator. Indeed, let

\[
Lf(t) = h(t)
\]

Then, if a unique inner product is defined, i.e.

\[
f(t) \neq g(t) \Rightarrow (k(t, s) \cdot f(t)) \neq (k(t, s) \cdot g(t))
\]

that is if, in principle, the integral transform is invertible, one can proceed as follows: Let \( k \) be an eigenvector for the adjoint operator \( \hat{L} \):

\[
\hat{L}k = \lambda k
\]

The original problem is transformed to a space with an independent variable \( \lambda \) by taking the inner products of the original equation and of the adjoint eigenvector equation with \( k(t, \lambda) \):

\[
(k \cdot Lf) = (k \cdot h)
\]

\[
(\hat{L}k \cdot f) = \lambda (k \cdot f)
\]

Now, from the definition of adjoint operators,

\[
(\hat{L}k \cdot f) = (k \cdot Lf) = (k \cdot h) = \lambda (k \cdot f)
\]

and therefore, in integral transform notation,

\[
\tilde{f} = \frac{h}{\lambda}
\]

So a (possibly complicated) linear operator equation has been transformed into a simple algebraic problem. Of course the solution for this problem is really known only if we have a procedure for inverting the transform.
3 INTRODUCTION TO FINITE TRANSFORMS

In the case where the inner product in the vector space under consideration can be defined as a (finite or infinite) sum rather than as an integral, i.e. when a basis for the space is known and all vectors can be written as linear combinations of the basis vectors, we obtain finite transforms.

Finite transforms are useful in boundary value problems with a denumerable set of eigenvalues, $\lambda_n$. These eigenvalues (or simply the index $n$) will play then the role of the transform variable $s$; so, finite transforms are defined in terms of a discrete variable (but they should not be confused with the so-called discrete transforms, such as the $z$-transform).

Consider the (Hilbert) space of square integrable functions $L^2$ and a maximal orthogonal (i.e. basis) set of functions $\{\phi_n\}$ in it. Then, the best approximation of any vector $f(x)$ in terms of the $\{\phi_n\}$ in $L^2$ is given by the generalized Fourier series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{(f \cdot \phi_n)}{\| \phi_n \|^2} \phi_n(x)$$

or, up to the $n$-th term, by

$$f(x) \simeq \sum_{n} f_n(x) = \sum_{n} \frac{(f \cdot \phi_n)}{\| \phi_n \|^2} \phi_n(x)$$

If the $\{\phi_n\}$ are eigenvectors of a Sturm-Liouville operator $L$, i.e.

$$L\phi_n = \lambda_n \phi_n$$

for

$$Lf = h(x)$$

with boundary conditions such that $L$ remains self-adjoint, then, using $\phi_n$ as kernel functions of the integral transform

$$(Lf \cdot \phi_n) = (h \cdot \phi_n) = \tilde{h}_n$$

$$(f \cdot L\phi_n) = (h \cdot \phi_n) = \tilde{h}_n$$

and

$$(f \cdot L\phi_n) = \lambda_n (f \cdot \phi_n) = \lambda_n \tilde{f}_n$$

Therefore

$$\tilde{f}_n = \frac{\tilde{h}_n}{\lambda_n}$$

and the relation

$$f(x) = \sum_{n=1}^{\infty} \frac{\tilde{h}_n}{\lambda_n \| \phi_n \|^2} \phi_n(x)$$

essentially provides the inversion of the transform.

The simplicity of the above derivation and the compact form of the last relation should not be misleading: in fact this last equation “codifies” the solution of a vast number of linear boundary value problems in transport phenomena (conduction, diffusion, dispersion) as well as in other fields of science and engineering.
4 INTRODUCTION TO INFINITE TRANSFORMS

When the inner product is defined in terms of a definite integral rather than a sum, we obtain an infinite transform:

\[ \tilde{f}(s) = (K(x, s) \cdot f(x)) = \int_a^b r(x)K(x, s)f(x) \, dx \]

where \( s \) (taking the place of \( \lambda \)) is now a continuous scalar variable. \( r(x) \) is a weight factor for the inner product (often 1) and \( K(x, s) \) is the kernel of the infinite transform.

Note that although the vectors (= functions) considered here are functions of one variable, this is done for convenience only. Extensions to multivariable and multidimensional situations are straightforward.

In order for the transform to be invertible, there should exist functions \( r^{(i)}(s) \) and \( K^{(i)}(s, x) \) such that

\[ f(x) = \int_c^d r^{(i)}(s)K^{(i)}(s, x)\tilde{f}(s) \, ds \]

for all \( f(x) \) in the space of interest.

Usually the limits on the (direct and inverse) transform integrals are either \((-\infty, \infty)\) or \((0, \infty)\).

To determine the kernel for an infinite transform of the operator \( L \) for which

\[ Lf(t) = h(t) \]

we usually are less restrictive than for finite transforms and seek a function \( k = K(x, s) \) such that

\[ \hat{L}k = s^k k + (\text{algebraic polynomial in } s) \]

4.1 The Laplace Transform

The Laplace transform of a function \( f(x) \) for \( x \in (0, \infty) \), denoted by \( Lf \) or \( \tilde{f}(s) \), is defined by choosing

\[ K(x, s) = e^{-sx}, \quad \text{Re}(s) > 0, \quad r(x) = 1 \]

i.e.

\[ Lf(x) = \tilde{f}(s) = \int_0^\infty e^{-sx} f(x) \, dx \]

Note that for this transform to exist it is required that \( f(x) \) is (i) piecewise smooth over every finite interval and that, (ii) it is of exponential order, i.e. there exist real constants \( K, c, x_0 \) such that

\[ |f(x)| < Ke^{ct} \quad \forall x > x_0 \]

The Laplace transform of \( f \) then exists (at least for \( s > c \)).

The inverse of the Laplace transform is denoted by \( L^{-1} \):

\[ f(x) = L^{-1}f(s) \]

For most practical purposes the Laplace transform pairs can be found directly in tables.

\(^1\)Typically, in engineering textbooks the original function is denoted by \( F(t) \) and its Laplace transform by \( f(s) \) or vice versa (Kreyszig).
4.1.1 Some Basic Properties of the Laplace Transform

- **Linearity:**
  \[ L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\} \equiv \alpha \tilde{f}(s) + \beta \tilde{g}(s) \]

- **Transform of the derivative of \( n \)-th order:**
  \[ L\{f^{(n)}(t)\} = s^n \tilde{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0) \]

- **Transform of the definite integral:**
  \[ L\left\{ \int_0^t f(\tau)d\tau \right\} = \frac{1}{s} \tilde{f}(s), \quad L^{-1}\left\{ \frac{1}{s} \tilde{f}(s) \right\} = \int_0^t f(\tau)d\tau \]

- **Shifting:**
  \[ L\{e^{\alpha t} f(t)\} = \tilde{f}(s - \alpha), \quad L^{-1}\left\{ \tilde{f}(s - \alpha) \right\} = e^{\alpha t} f(t) \]

- **Differentiation of the Laplace Transform:**
  \[ L\{tf(t)\} = -\tilde{f}'(s), \quad L^{-1}\left\{ -\tilde{f}'(s) \right\} = -tf(t) \]

- **Integration of the Laplace Transform:**
  \[ L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \tilde{f}(z)dz, \quad L^{-1}\left\{ \int_s^\infty \tilde{f}(z)dz \right\} = \frac{f(t)}{t} \]

- **Convolution:**
  \[ \tilde{h}(s) = \tilde{f}(s)\tilde{g}(s) \Rightarrow h(t) = \int_0^t f(\tau)g(t-\tau)d\tau \equiv (f \ast g)(t) \]

Details on the above as well as additional properties of the Laplace Transform, and tables of transform pairs, can be found in almost any applied mathematics text (e.g., Kreyszig, Chapter 6). Note that although general inversion relations exist they are not trivial to implement and the practicality of using Laplace Transform methods relies on the availability of extensive tables of transform pairs: Recall the usefulness of the Laplace Transform properties in transforming ordinary differential equations to the corresponding **subsidiary** algebraic equations, solving those equations in the \( s \)-domain and then transforming the solution into the \( t \)-domain.

### 4.2 The Fourier Transform

The Fourier transform of a function \( f(x) \) for \( x \in (-\infty, \infty) \), denoted by \( Ff \) is defined by choosing

\[ K(x, s) = e^{-ixs}, \quad r(x) = 1 \]

i.e.

\[ Ff(x) = \tilde{f}(s) = \int_{-\infty}^{\infty} e^{-ixs} f(x) dx \]

The inverse of the Fourier transform is denoted by \( F^{-1} \):

\[ f(x) = F^{-1} f(s) \]
The Fourier transform is more symmetric than the Laplace transform and calculating its inverse is no more complicated than calculating the original transform (in practice, tabulated results are often adequate):

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixs} \tilde{f}(s) \, ds \]

Note, that to retain full symmetry, the following alternative definitions of the Fourier transform (and the corresponding calculation of its inverse) are also often used:

\[ Ff(x) = \tilde{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(x) \, dx \]

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \tilde{f}(s) \, ds \]

5 STUDY ASSIGNMENT

Review thoroughly the material from Chapter 6 (Laplace Transforms) of Kreyszig. Also study Sections 5.8 and 5.9 of Chapter 5 (Sturm-Liouville Problems and Eigenfunction Expansions) that were covered from a somewhat more advanced and general viewpoint in our Lecture Units # 6 and 7.

Also (see Homework Assignment # 4) review those sections of Chapters 1 and 2 of Kreyszig that you feel you might have forgotten. This – as well as most of the material of Chapters 3 and 4 – is typical undergraduate material for ordinary differential equations.

6 PROBLEMS FOR THE MATERIAL OF LECTURE UNIT 8

A problem set covering material of this lecture unit (as well as material from Lecture Units 6 and 7) is assigned with this Lecture Unit.